

The Big Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent.

- ① A is invertible
- ② The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- ③ The reduced row-echelon form of A is the $n \times n$ identity matrix I_n .
- ④ The system $A\vec{x} = \vec{b}$ is solvable for every \vec{b} . Moreover, there is exactly one solution.
- ⑤ $\det(A) \neq 0$
- ⑥ The rows of A are linearly independent
- ⑦ The columns of A are linearly independent.
- ⑧ A has rank n .
- ⑨ $\lambda = 0$ is not an eigenvalue of A

What does "the following are equivalent" mean?
It means that if one of ①-⑨ is true, then all of ①-⑨ are true. And if any of ①-⑨ are false, then all of ①-⑨ are false.