Math 3450 - Homework # 2
Set Theory

1. Let $A = \{1, 5, -12, 100, 1/3, \pi\}$, $B = \{5, 1, -12, 18, -1/3\}$, $C = \{10, -1, 0\}$, $D = \{1, 2\}$, and $E = \{1, -1\}$. Calculate the following:

   (a) $A \cup B$
   Solution: $\{1, 5, -12, 100, 1/3, \pi, 18, -1/3\}$

   (b) $A \cap B$
   Solution: $\{1, 5, -12\}$

   (c) $A \cap C$
   Solution: $\emptyset$

   (d) $A \cap \emptyset$
   Solution: $\emptyset$

   (e) $B \cup \emptyset$
   Solution: B

   (f) $D \times E$
   Solution: $\{(1, 1), (1, -1), (2, 1), (2, -1)\}$

   (g) $(D \cap A) \times (E \cup D)$
   Solution: $D \cap A = \{1\}$, $E \cup D = \{1, 2, -1\}$, $(D \cap A) \times (E \cup D) = \{(1, 1), (1, 2), (1, -1)\}$

   (h) $C \times D$
   Solution: $\{(10, 1), (-1, 1), (0, 1), (10, 2), (-1, 2), (0, 2)\}$

   (i) $A - B$
   Solution: $\{100, 1/3, \pi\}$

   (j) $C - A$
   Solution: C

   (k) $A - \emptyset$
   Solution: A

2. Let $A = \{2k \mid k \in \mathbb{Z}\}$ and $B = \{3n \mid n \in \mathbb{Z}\}$. Prove that $A \cap B = \{6m \mid m \in \mathbb{Z}\}$.

Proof. **(⊆)**

First we show that $A \cap B \subseteq \{6m \,|\, m \in \mathbb{Z}\}$.

Suppose that $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

Then $x = 2k$ and $x = 3n$ where $k, n \in \mathbb{Z}$.

Thus $2k = 3n$.

Therefore, $3n$ is even.

Since an odd integer multiplied by and odd integer is odd, we cannot have that n is odd.

Therefore n is even.

So $n = 2l$ where $l \in \mathbb{Z}$.

Thus $x = 3n = 3(2l) = 6l \in \{6m \,|\, m \in \mathbb{Z}\}$.

So $A \cap B \subseteq \{6m \,|\, m \in \mathbb{Z}\}$.

(⊇)

Now we show that $\{6m \,|\, m \in \mathbb{Z}\} \subseteq A \cap B$.

Let $x \in \{6m \,|\, m \in \mathbb{Z}\}$.

Then $x = 6m$ where $m \in \mathbb{Z}$.

Note that $x = 6m = 2(3m) = 3(2m)$.

Hence $x \in A$ and $x \in B$.

Thus $x \in A \cap B$.

So $\{6m \,|\, m \in \mathbb{Z}\} \subseteq A \cap B$.

Therefore by **(⊆)** and **(⊇)** we get that $A \cap B = \{6m \,|\, m \in \mathbb{Z}\}$. □

3. Let A, B, and C be sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.

Proof. Let $x \in A - C$.

We will show that $x \in B - C$.

We know that $x \in A$ and $x \notin C$, because $x \in A - C$.

Since $x \in A$ and $A \subseteq B$ we have that $x \in B$.

Since $x \in B$ and $x \notin C$ it follows that $x \in B - C$.

Therefore $A - C \subseteq B - C$. □

4. Let A and B be sets. Prove that $A \subseteq B$ if and only if $A - B = \emptyset$.

Proof 1 - by contraposition. In this version of the proof we will use contraposition. Recall that P iff Q is equivalent to $\neg P$ iff $\neg Q$. Thus "$A \subseteq B$ if and only if $A - B = \emptyset$" is equivalent to "$A \not\subseteq B$ if and only if $A - B \neq \emptyset$". We instead prove this second statement.

(⇒) Suppose that $A \not\subseteq B$.

This means that there exists an $x \in A$ with $x \notin B$.

Thus there exists x with $x \in A - B$.

So $A - B \neq \emptyset$.

(⇐) Suppose that $A - B \neq \emptyset$.

Then there exists $x \in A - B$.

So $x \in A$ and $x \notin B$.

Thus $A \not\subseteq B$.

□

Proof 2 - by contradiction. **(⇒)**

First, we will show that if $A \subseteq B$, then $A - B = \emptyset$.

We will prove this by contradiction.

Suppose that $A \subseteq B$, but $A - B \neq \emptyset$.

Then there exists $x \in A - B$.

So $x \in A$ and $x \notin B$.

But $A \subseteq B$, so $x \in A$ implies that $x \in B$.

Contradiction.

Therefore $A - B = \emptyset$.

(⇐)

Next, we will show that if $A - B = \emptyset$, then $A \subseteq B$.

Suppose $x \in A$. We will show that $x \in B$.

Suppose to the contrary that $x \notin B$.

Then $x \in A - B$, since $x \in A$ and $x \notin B$.

But $A - B = \emptyset$.

Contradiction.

Therefore $x \in B$.

Therefore $A \subseteq B$. □

5. Let A, B, and C be sets. Prove that if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Proof. Suppose $x \in A \cup C$.

We will show that $x \in B \cup C$.

We know that $x \in A$ or $x \in C$.

Case 1: Suppose that $x \in A$.

Since $A \subseteq B$ we have that $x \in B$.

Thus $x \in B$ and $x \in C$.

So $x \in B \cup C$.

Case 2: Suppose that $x \in C$.

Then $x \in B \cup C$.

In either case above, we get that $x \in B \cup C$.

So $A \cup C \subseteq B \cup C$. □

6. Let A, B, and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. **(⊆)**

First, we will show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Suppose that $(x, y) \in A \times (B \cap C)$.

Then $x \in A$ and $y \in B \cap C$.

Since $y \in B \cap C$, we have that $y \in B$ and $y \in C$.

Since $x \in A$ and $y \in B$, we have that $(x, y) \in A \times B$.

Since $x \in A$ and $y \in C$, we have that $(x, y) \in A \times C$.

So $(x, y) \in (A \times B) \cap (A \times C)$.

Therefore $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

($\supseteq$)

Next, we will show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Suppose that $(x, y) \in (A \times B) \cap (A \times C)$.

Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$.

Since $(x, y) \in A \times B$ we get that $x \in A$ and $y \in B$.

Since $(x, y) \in A \times C$ we get that $x \in A$ and $y \in C$.

So $y \in B \cap C$, because $y \in B$ and $y \in C$.

Thus $(x, y) \in A \times (B \cap C)$, because $x \in A$ and $y \in B \cap C$.

Ergo, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Therefore by **($\subseteq$)** and **($\supseteq$)** we get that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$

7. Let A, B, and C be sets. Prove or disprove: If $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, then $A \cap C \neq \emptyset$.

   Solution:

   False. Here's a counterexample: $A = \{1\}, B = \{1, 2\}, C = \{2\}$.

8. Let $A_n = \{x \in \mathbb{Z} \mid -n \leq x \leq n\}$. List the elements in the sets A_1, A_2, A_3, and A_4. Then calculate the following sets $\bigcap_{i=2}^{\infty} A_n$ and $\bigcup_{i=5}^{\infty} A_n$.

   Solution:

   $A_1 = \{-1, 0, 1\}$, $A_2 = \{-2, -1, 0, 1, 2\}$, $A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$, $A_4 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

   $\bigcap_{i=2}^{\infty} A_n = \{-2, -1, 0, 1, 2\}$

   $\bigcup_{i=5}^{\infty} A_n = \mathbb{Z}$

9. Calculate the following intersections and unions.

   (a) Calculate $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ where $A_n = (-n, n)$.

   Solution:

$\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$
$\bigcap_{n=1}^{\infty} A_n = (-1, 1)$

(b) Calculate $\bigcup_{n=2}^{\infty} A_n$ and $\bigcap_{n=2}^{\infty} A_n$ where $A_n = (1/n, 1)$.

Solution:
$\bigcup_{n=2}^{\infty} A_n = (0, 1)$
$\bigcap_{n=2}^{\infty} A_n = (1/2, 1)$

(c) Calculate $\bigcup_{n=3}^{\infty} A_n$ and $\bigcap_{n=3}^{\infty} A_n$ where $A_n = (2 + 1/n, n)$.

Solution:
$\bigcup_{n=3}^{\infty} A_n = (2, \infty)$
$\bigcap_{n=3}^{\infty} A_n = (2 + 1/3, 3) = (7/3, 3)$

10. Let A, B, and C be sets. Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof. **(⊆)** First, we will show that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
Suppose $x \in A \cap (B \cap C)$.
Then $x \in A$ and $x \in B \cap C$.
So $x \in A$ and $x \in B$ and $x \in C$.
Since $x \in A$ and $x \in B$ we have that $x \in A \cap B$.
So $x \in (A \cap B) \cap C$, because $x \in A \cap B$ and $x \in C$.
Therefore, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
(⊇) Now we will show that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.
Let $x \in (A \cap B) \cap C$.
Then $x \in (A \cap B)$ and $x \in C$.
Thus $x \in A$ and $x \in B$ and $x \in C$.
Since $x \in B$ and $x \in C$ we have that $x \in B \cap C$.
Hence $x \in A \cap (B \cap C)$ since $x \in A$ and $x \in B \cap C$.

Therefore, by **(⊆)** and **(⊇)** we get that $A \cap (B \cap C) = (A \cap B) \cap C$. □

11. Let A, B, and C be sets. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. **(⊆)** First, we will show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Let $x \in A \cup (B \cap C)$.

We know $x \in A$ or $x \in B \cap C$.

Case 1: Suppose that $x \in A$.

Then $x \in A \cup B$, since $x \in A$.

Also, $x \in A \cup C$, since $x \in A$.

Thus $x \in A \cup B$ and $x \in A \cup C$.

So, $x \in (A \cup B) \cap (A \cup C)$.

Case 2: Suppose that $x \in B \cap C$.

Then $x \in B$ and $x \in C$.

So $x \in A \cup B$, because $x \in B$.

Also $x \in A \cup C$, because $x \in C$.

Thus $x \in A \cup B$ and $x \in A \cup C$.

So $x \in (A \cup B) \cap (A \cup C)$.

In either case, we have $x \in (A \cup B) \cap (A \cup C)$.

So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

(⊇) Next, we will show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Suppose that $x \in (A \cup B) \cap (A \cup C)$.

Then $x \in (A \cup B)$ and $x \in (A \cup C)$.

So $x \in A$ or $x \in B$, because $x \in (A \cup B)$.

Case 1: Suppose that $x \in A$.

Then $x \in A \cup (B \cap C)$, because $x \in A$.

Case 2: Suppose that $x \in B$.

We know that $x \in A$ or $x \in C$, because $x \in (A \cup C)$ (from above before case 1).

We break case 2 into two sub-cases.

Case 2i: Suppose that $x \in A$.

Then $x \in A \cup (B \cap C)$, because $x \in A$.

Case 2ii: Suppose that $x \in C$.

Then $x \in B \cap C$, because $x \in B$ and $x \in C$.

So $x \in A \cup (B \cap C)$, because $x \in B \cap C$.

In every case, we have $x \in A \cup (B \cap C)$.

Therefore $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Therefore, by **(⊆)** and **(⊇)** we get that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ □

12. Let A, B, and C be sets. Prove that if $A \subseteq B$ then $A \subseteq B \cup C$.

    Solution: Suppose that $A \subseteq B$.

    We use this to show that $A \subseteq B \cup C$.

    Let $x \in A$.

    Since $A \subseteq B$ and $x \in A$, we know that $x \in B$.

    Since $x \in B$, we know that $x \in B \cup C$.

    Therefore, if $x \in A$, then $x \in B \cup C$ is true.

    So $A \subseteq B \cup C$.

13. Let $A = \{1, x, 5\}$. List the elements of the power set $\mathcal{P}(A)$.

    Solution:

    $\emptyset, \{1\}, \{x\}, \{5\}, \{1, x\}, \{1, 5\}, \{x, 5\}, A$

14. Let A and B be sets.

    (a) Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

    Proof. **(⊆)** First, we will show that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
    Suppose that $S \in \mathcal{P}(A \cap B)$. We will show that $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
    We know that $S \subseteq A \cap B$, because $S \in \mathcal{P}(A \cap B)$.
    So every element of S is in $A \cap B$.

So every element of S is in both A and B.
So $S \subseteq A$ and $S \subseteq B$.
So $S \in \mathcal{P}(A)$ and $\mathcal{P}(B)$.
So $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
Therefore $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

($\supseteq$) Next, we will show that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.
Suppose that $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. We will show that $S \in \mathcal{P}(A \cap B)$.
We know that $S \in \mathcal{P}(A)$ and $\mathcal{P}(B)$, because $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
So $S \subseteq A$ and $S \subseteq B$.
So every element of S is in both A and B.
So every element of S is in $A \cap B$.
So $S \subseteq A \cap B$.
So $S \in \mathcal{P}(A \cap B)$.
Therefore $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Therefore, by **($\subseteq$)** and **($\supseteq$)** we get that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. □

(b) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Suppose that $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
Then $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$.
Case 1: Suppose that $S \in \mathcal{P}(A)$.
Then $S \subseteq A$.
So $S \subseteq A \cup B$, by problem 12 above.
Case 2: $S \in \mathcal{P}(B)$
Then $S \subseteq B$.
So $S \subseteq A \cup B$, by problem 12 above.

In either case, we have $S \subseteq A \cup B$.
So $S \in \mathcal{P}(A \cup B)$.
Thus, if $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $S \in \mathcal{P}(A \cup B)$.
Therefore $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

(c) Give an example where $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Solution:
$A = \{1\}, B = \{2\}$
$\mathcal{P}(A) = \{\emptyset, \{1\}\}$
$\mathcal{P}(B) = \{\emptyset, \{2\}\}$
$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$
$A \cup B = \{1, 2\}$
$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

15. Let A and B be sets. Prove that $A - B$ and B are disjoint.

Proof. We will show that $(A - B) \cap B = \emptyset$.

We do this by contradiction.

Suppose that $(A - B) \cap B \neq \emptyset$.

Then there exists $x \in (A - B) \cap B$.

So $x \in A - B$ and $x \in B$.

But $x \in A - B$ implies that $x \in A$ and $x \notin B$.

Thus we have that $x \in B$ and $x \notin B$.

Contradiction. (We cannot have both $x \in B$ and $x \notin B$.)

Therefore $(A - B) \cap B = \emptyset$.

Therefore $A - B$ and B are disjoint. □

16. Let A, B, C, and D be sets. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. **(⊆)** First, we will show that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

Suppose $(x, y) \in (A \times B) \cap (C \times D)$.

Then $(x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$.

So $x \in A$ and $y \in B$, because $(x, y) \in (A \times B)$.

Also, $x \in C$ and $y \in D$, because $(x, y) \in (C \times D)$.

So $x \in A \cap C$, because $x \in A$ and $x \in C$.

Also $y \in B \cap D$, because $y \in B$ and $y \in D$.

So $(x, y) \in (A \cap C) \times (B \cap D)$, because $x \in A \cap C$ and $y \in B \cap D$.

Therefore $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

($\supseteq$) Next, we will show that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$.

Suppose that $(x, y) \in (A \cap C) \times (B \cap D)$.

Then $x \in A \cap C$ and $y \in B \cap D$.

So $x \in A$ and $x \in C$, because $x \in A \cap C$.

Also $y \in B$ and $y \in D$, because $y \in B \cap D$.

So $(x, y) \in A \times B$, because $x \in A$ and $y \in B$.

Also, $(x, y) \in C \times D$, because $x \in C$ and $y \in D$.

Therefore $(x, y) \in (A \times B) \cap (C \times D)$, because $(x, y) \in A \times B$ and $(x, y) \in C \times D$.

So $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$.

Therefore, by **($\subseteq$)** and **($\supseteq$)** we get that$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. $\square$