Math 3450 - Homework # 3
Equivalence Relations and Well-Defined Operations

1. A set S and a relation $\sim$ on S is given. For each example, check if $\sim$ is (i) reflexive, (ii) symmetric, and/or (iii) transitive. If $\sim$ satisfies the property that you are checking, then prove it. If $\sim$ does not satisfy the property that you are checking, then give an example to show it.

   (a) $S = \mathbb{R}$ where $a \sim b$ if and only if $a \leq b$.

   (b) $S = \mathbb{R}$ where $a \sim b$ if and only if $|a| = |b|$.

   (c) $S = \mathbb{Z}$ where $a \sim b$ if and only if $a|b$.

   (d) S is the set of subsets of $\mathbb{N}$ where $A \sim B$ if and only if $A \subseteq B$. Some examples of elements of S are $\{1, 10, 199\}$, $\{2, 7, 10\}$, and $\{2, 10, 3, 7\}$. Note that $\{2, 7, 10\} \sim \{2, 10, 3, 7\}$

2. Consider the set $S = \mathbb{R}$ where $x \sim y$ if and only if $x^2 = y^2$.

   (a) Find all the numbers that are related to $x = 1$. Repeat this exercise for $x = \sqrt{2}$ and $x = 0$.

   (b) Prove that $\sim$ is an equivalence relation on S.

   (c) Draw a number line. Draw a picture of the equivalence class of 1. Repeat this for $x = 0$, $x = \sqrt{6}$, $x = -3$.

   (d) Describe the elements of $S/\sim$.

3. Consider the set $S = \mathbb{Z}$ where $x \sim y$ if and only if $2|(x + y)$.

   (a) List six numbers that are related to $x = 2$.

   (b) Prove that $\sim$ is an equivalence relation on S.

   (c) Draw a picture of the set of integers. Next, circle the numbers that are in the equivalence class of -3.

   (d) Describe the elements of $S/\sim$. Draw a picture of several equivalence classes.

4. Show that the operation $\overline{a} \oplus \overline{b} = \overline{a}^2 + \overline{b}^2$ is a well-defined operation for $\mathbb{Z}_n$. Here $\overline{a}^2$ means $\overline{a} \cdot \overline{a}$. For example, in $\mathbb{Z}_4$ we have that

$$\overline{2} \oplus \overline{3} = \overline{2} \cdot \overline{2} + \overline{3} \cdot \overline{3} = \overline{4} + \overline{9} = \overline{1}.$$

5. Given two integers a and b, let $\min(a, b)$ denote the minimum (smaller) of a and b. Let n be an integer with $n \geq 2$. Is the operation $\overline{a} \oplus \overline{b} = \overline{\min(a, b)}$ a well-defined operation on $\mathbb{Z}_n$?

6. (a) Show that the operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc}$ is not a well-defined operation on $\mathbb{Q}$. (b) Is the operation well-defined on $\mathbb{Q} - \{0\}$?

7. Is the operation $\overline{a} \oplus \overline{b} = \overline{a^b}$ a well-defined operation on $\mathbb{Z}_n$?

8. (Constructing the integers from the natural numbers) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $a+d = b+c$.

   (a) Is $(3, 6) \sim (7, 10)$?

   (b) Is $(1, 1) \sim (3, 5)$?

   (c) Prove that $\sim$ is an equivalence relation.

   (d) List five elements from each of the following equivalence classes: $\overline{(1, 1)}$, $\overline{(1, 2)}$, $\overline{(5, 12)}$.

   (e) Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a + c, b + d)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

9. (Constructing the rational numbers from the integers) Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $ad = bc$.

   (a) Is $(1, 5) \sim (-3, -15)$?

   (b) Is $(-1, 1) \sim (2, 3)$?

   (c) Prove that $\sim$ is an equivalence relation.

   (d) List five elements from each of the following equivalence classes: $\overline{(1, 1)}$, $\overline{(0, 2)}$, $\overline{(2, 3)}$.

   (e) Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(ad + bc, bd)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

   (f) Define the operation $\overline{(a, b)} \odot \overline{(c, d)} = \overline{(ac, bd)}$. Prove that $\odot$ is well-defined on the set of equivalence classes.