Math 3450 - Homework # 4
Functions

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{7, 8, -1, \pi, 1/2\}$.

   (a) Give an example of a function $f : A \to B$ that is one-to-one.

   (b) Give an example of a function $f : A \to B$ that is not one-to-one.

   (c) Give an example of a function $f : B \to A$ that is onto.

   (d) Give an example of a function $f : B \to A$ that is not onto.

2. Consider the following functions. For each function f, (i) either prove that f is one-to-one or give an example to show otherwise, and (ii) either prove that f is onto, or give an example to show otherwise. (iii) If f is a bijection, find a formula for f^{-1}.

   You will need the following definition. Let $M_2(\mathbb{R})$ be the set of all 2×2 matrices with entries from the real numbers. That is,

   $$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

   For example, $\begin{pmatrix} 1 & 0 \\ -10 & \pi \end{pmatrix}$ is an element of $M_2(\mathbb{R})$.

   (a) Let $f : \mathbb{Z} \to A$ given by $f(k) = 2k$ where $A = \{2n \mid n \in \mathbb{Z}\}$. (For example, $f(7) = 2 \cdot 7 = 14$. Note: The set A is commonly referred to as $2\mathbb{Z}$.)

   (b) $f : \mathbb{Q} \to \mathbb{Q}$ where $f(x) = x^3$.

   (c) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 5$.

   (d) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^4 - 16$.

   (e) $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ given by $f(\overline{x}) = \overline{2} \cdot \overline{x} + \overline{1}$.

   (f) $f : M_2(\mathbb{R}) \to \mathbb{R}$ where $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$.

   (g) $f : M_2(\mathbb{R}) \to \mathbb{R}$ where $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$.

3. Let $A = \{1, 2, 3, 4\}$. Let $i_A : A \to A$ be the identity function on A. That is, $i_A(x) = x$ for all $x \in A$.

(a) Let $f : A \to A$ where $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, and $f(4) = 4$. Draw a picture of f. Draw a picture of f^{-1}. Show that $f \circ f^{-1} = i_A$ and $f^{-1} \circ f = i_A$.

(b) Let $g : A \to A$ where $g(1) = 1$, $g(2) = 3$, $g(3) = 4$, and $g(4) = 2$. Draw a picture of g. Draw a picture of g^{-1}. Show that $g \circ g^{-1} = i_A$ and $g^{-1} \circ g = i_A$.

4. Let a and n be integers with $n \geq 2$. Define $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$ by $f_a(\overline{x}) = \overline{a} \cdot \overline{x}$.

   (a) Prove that f_a is a well-defined function.

   (b) Draw a picture of f_4 when $n = 6$.

   (c) Draw a picture of f_2 when $n = 3$.

   (d) Prove that $f_c \circ f_d = f_{cd}$.

   (e) Prove that $f_{cd} = f_{dc}$ for all integers c and d.

   (f) Prove: If $y \equiv w (\text{mod } n)$, then $f_y = f_w$.

   (g) Prove that if $\gcd(a, n) > 1$, then f_a is not a bijection. [Hint: Note that $f_a(\overline{0}) = \overline{0}$. Find $\overline{k} \neq \overline{0}$ with $f_a(\overline{k}) = \overline{0}$.]

   (h) Consider $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_5$. Find f_3^{-1} and express it in the form f_b for some integer b.

5. Consider the function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f(\overline{x}) = \overline{x}^2$.

   (a) Prove that f a well-defined function.

   (b) Draw a picture of f when $n = 5$.

   (c) Draw a picture of f when $n = 6$.

   (d) Prove that if $n > 2$ then f is not one-to-one.

6. Let $f : \mathbb{Q} \to \mathbb{Z}$ be defined by $f(m/n) = m$. For example, $f(2/9) = 2$ and $f(5/10) = 5$. Is f a well-defined function? If so prove it. If not explain why not.

7. Let n be an integer with $n \geq 2$. Let a be an integer. Define $g_a : \mathbb{Z}_n \to \mathbb{Z}_n$ by the formula $g_a(\overline{x}) = \overline{x} + \overline{a}$.

   (a) Prove that g_a is well-defined.

   (b) Draw a picture of g_3 and g_2 when $n = 4$.

(c) Compute and draw a picture of $g_3 \circ g_2$ and $g_2 \circ g_3$ when $n = 4$.

(d) Prove that g_a is a bijection for any n.

(e) Find a formula for g_a^{-1}.

8. Give an example of $f : A \to B$ and $g : B \to C$ where the following are true:

   (a) f is not onto, but $g \circ f$ is onto.

   (b) g is not one-to-one, but $g \circ f$ is one-to-one.

9. Suppose that $f : A \to B$ and $g : B \to C$. Prove: If f is not one-to-one, then $g \circ f$ is not one-to-one.

10. Suppose that $f : A \to B$ and $g : B \to C$. Prove: If g is not onto, then $g \circ f$ is not onto.

11. Let $n \geq 2$ be an integer. Consider the reduction mod n map $\pi_n : \mathbb{Z} \to \mathbb{Z}_n$ given by the formula $\pi_n(x) = \overline{x}$.

    For example, $\pi_6(2) = \overline{2}$ and $\pi_6(18) = \overline{18} = \overline{0}$ since $18 \equiv 0 (\text{mod } 6)$.

    (a) Calculate $\pi_6(-1)$, $\pi_6(10)$, $\pi_6(7)$, and $\pi_6(-17)$. Draw a picture of the π_6 map. Is π_6 one-to-one? Is π_6 onto?

    (b) Let $X = \{1, 17, -5, 102, -13\}$. Calculate $\pi_6(X)$.

    (c) Let $Y = \{\overline{0}\}$. Prove that $\pi_6^{-1}(Y) = \{6k \mid k \in \mathbb{Z}\}$.

    (d) Let $Y = \{\overline{1}\}$. Prove that $\pi_6^{-1}(Y) = \{6k + 1 \mid k \in \mathbb{Z}\}$.

    (e) What is $\pi_6^{-1}(\{\overline{0}, \overline{3}\})$ equal to? Prove your answer.

12. Let $A = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$. Let $f : A \times A \to A$ where $f(m, n) = m^2 + n^2$.

    (a) Calculate $f(3, 5)$, $f(1, 1)$, and $f(2, 1)$.

    (b) Let $C = \{(0, 0), (1, 10), (2, 5)\}$. Calculate $f(C)$.

    (c) Let $B = \{1, 2, 3, 4\}$. Find $f^{-1}(B)$.

    (d) Show that f is not one-to-one.

    (e) Show that f is not onto.

13. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 2$.

(a) $f([0,1])$

(b) $f^{-1}([0,1))$

(c) $f^{-1}([-3,-1))$

14. Suppose that X, Y, W, Z, A, B are sets. Let $f : X \to Y$, $W \subseteq X$, $Z \subseteq X$, $A \subseteq Y$, and $B \subseteq Y$.

    (a) Prove that $f(W \cup Z) = f(W) \cup f(Z)$.

    (b) Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

    (c) Prove that $X - f^{-1}(A) \subseteq f^{-1}(Y - A)$.

15. Let A be a set. Define the function $f : \mathcal{P}(A) \to \mathcal{P}(A)$ where $f(X) = A - X$ for any $X \subseteq A$.

    (a) Draw a picture of f when $A = \{1, 2, 3\}$.

    (b) If $X \subseteq A$, then $A - (A - X) = X$.

    (c) For general A prove that f is a bijection.

    (d) For general A prove that $f = f^{-1}$.