Math 446 - Homework # 3

1. Prove the following:

   (a) Given $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist $x, y \in \mathbb{Z}$ with $\gcd(x, y) = 1$ and $\frac{a}{b} = \frac{x}{y}$.

   (b) If p is a prime and a is a positive integer and $p|a^n$, then $p^n|a^n$.

   (c) $\sqrt[5]{5}$ is irrational.

   (d) If p is a prime, then $\sqrt{p}$ is irrational.

2. (a) Suppose that a, b, c are integers with $a \neq 0$ and $b \neq 0$. If $a|c$, $b|c$, and $\gcd(a, b) = 1$, then $ab|c$.

   (b) Prove that $\sqrt{6}$ is irrational.

3. Prove that $\log_{10}(2)$ is an irrational number.

4. We say that an integer $n \geq 2$ is a **perfect square** if $n = m^2$ for some integer $m \geq 2$. Prove that n is a perfect square if and only if the prime factorization of $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ has even exponents (that is, all the k_i are even).

5. (a) Let a and b be positive integers. Prove that $\gcd(a, b) > 1$ if and only if there is a prime p satisfying $p|a$ and $p|b$.

   (b) Let a, b, and n be positive integers. Prove that if $\gcd(a, b) > 1$ if and only if $\gcd(a^n, b^n) > 1$.

6. Suppose that x and y are positive integers where $4|xy$ but $4 \nmid x$. Prove that $2|y$.

7. Let a and b be positive integers. Suppose that 5 occurs in the prime factorization of a exactly four times and 5 occurs in the prime factorization of b exactly two times. How many times does 5 occur in the prime factorization of $a + b$?

8. A positive integer $n \geq 2$ is called **squarefree** if it is not divisible by any perfect square. For example, 12 is not squarefree because $4 = 2^2$ is a perfect square and $4|12$. However, 10 is squarefree. (Recall the definition of perfect square from problem 4.)

(a) Prove that a positive integer $n \geq 2$ is squarefree if and only if n can be written as the product of distinct primes.

(b) Express the number $32,955,000 = 2^3 \cdot 3 \cdot 5^4 \cdot 13^3$ as the product of a squarefree number and a perfect square.

(c) Let $n \geq 2$ be a positive integer. Then either n is squarefree, or n is a perfect square, or n is the product of a squarefree number and a perfect square.

9. Suppose that $x, y, z \in \mathbb{Z}$ such that $x > 0$, $y > 0$, $z > 0$, $\gcd(x, y, z) = 1$, and $x^2 + y^2 = z^2$. Prove that $\gcd(x, z) = 1$. [Hint: Use Exercise 5.]