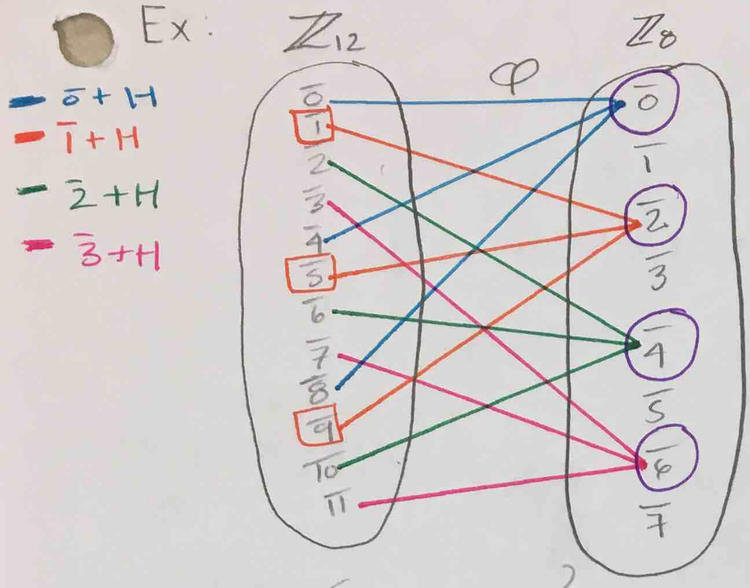
First Isomorphism Theorem

Ex:



- image of $\varphi = \varphi(\mathbb{Z}_{12}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$

$\ker(\varphi) = \{\bar{0}, \bar{4}, \bar{8}\} = H$

$H \trianglelefteq \mathbb{Z}_{12}$ since its the kernel of φ

so $\mathbb{Z}_{12}/H$ is a group.

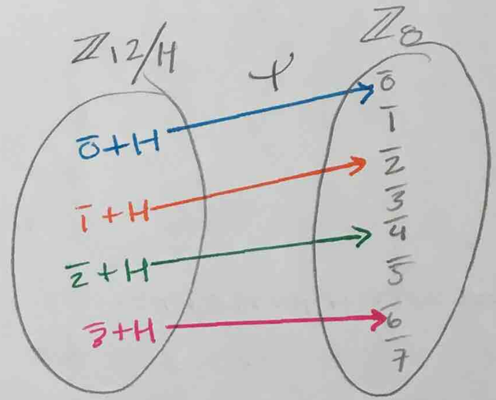
Let's calculate its element

$\bar{0} + H = \{\bar{0}, \bar{4}, \bar{8}\}$

$\bar{1} + H = \{\bar{1}, \bar{5}, \bar{9}\}$

$\bar{2} + H = \{\bar{2}, \bar{6}, \overline{10}\}$

$\bar{3} + H = \{\bar{3}, \bar{7}, \overline{11}\}$



Facts: ψ is homomorphism

ψ is 1-1

$\mathbb{Z}_{12}/H \cong \psi(\mathbb{Z}_8)$
$\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$

$*\ \mathrm{im}(\varphi) = \varphi(\mathbb{Z}_{12})$
$=$ Range

First Isomorphism Theorem

Let G and G' be groups and $\varphi: G \to G'$ be a homomorphism. Let $H = \ker(\varphi)$

Define $\psi: G/H \longrightarrow \mathrm{im}(\varphi)$ by $\psi(gH) = \varphi(g)$

then ψ is an isomorphism between G/H and $\mathrm{im}(\varphi)$ so, $G/\ker\varphi \cong \mathrm{im}(\varphi)$.

proof:

$H = \ker(\varphi)$ gives us that $H \trianglelefteq G$ and so G/H is a group.

ψ is well defined

suppose $g_1H = g_2H$ where $g_1, g_2 \in G$

we need to show that $\psi(g_1H) = \psi(g_2, H)$

since $g_1H = g_2H$ we know that $g_1 \in g_2H$

so $g_1 = g_2h$ where $h \in H$.

Then $\boxed{\psi(g_1H)} = \varphi(g_1) = \varphi(g_2h) = \varphi(g_2)\underbrace{\varphi(h)} = \varphi(g_2)\cdot e' = \varphi(g_2) \boxed{= \psi(g_2H)}$.

$h \in \ker(\varphi)$

$\varphi(h) = e' =$ identity

ψ is a homomorphism

Let $g_1H, g_2H \in G/H$, where $g_1, g_2 \in G$

Then $\psi((g_1H)(g_2H)) \underset{\uparrow}{=} \psi((g_1g_2)H) = \varphi(g_1g_2) \underset{\uparrow}{=} \varphi(g_1)\varphi(g_2)$

def of operation in G/H

φ is a homomorphism

$= \psi(g_1H)\psi(g_2H)$

ψ is 1-1

suppose $\psi(g_1H) = \psi(g_2H)$ where $g_1, g_2 \in G$

Then $\varphi(g_1) = \varphi(g_2)$ so, $\varphi(g_2)^{-1}\varphi(g_1) = \varphi(g_2)^{-1}\varphi(g_2)$

Then $\varphi(g_2^{-1})\varphi(g_1) = e'$ so, $\varphi(g_2^{-1}g_1) = e'$

Thus, $g_2^{-1}g_1 \in \ker(\varphi) = H \therefore g_1H = g_2H$

1-1

$aH = bH$

$b^{-1}a \in H$

$a \in bH$

automatically onto its range

thus ψ is an isomorphism between G/H and $\operatorname{im}(\psi) = \operatorname{im}(\varphi)$ ☒

11/30 Wednesday Week 15 November 30, 2016

Theorem: Let G be a group and $H \leq G$. Then the following are equivalent.

① $ghg^{-1} \in H \ \forall g \in G$ and $h \in H$

② $gHg^{-1} = H \ \forall g \in G$ where $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

③ H is normal, that is $gH = Hg \ \forall g \in G$

Example: $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

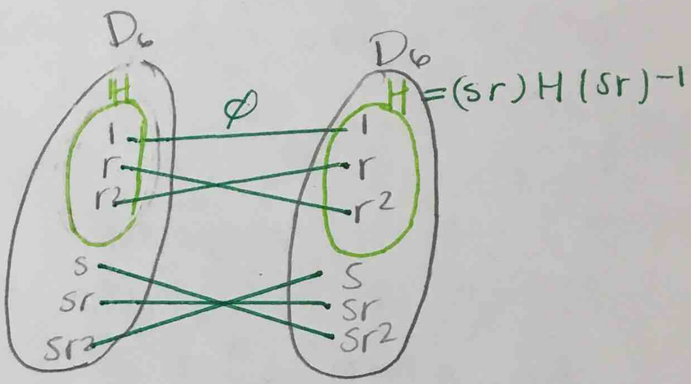
$H = \{1, r, r^2\}$

$H \trianglelefteq D_6$

$g = sr$

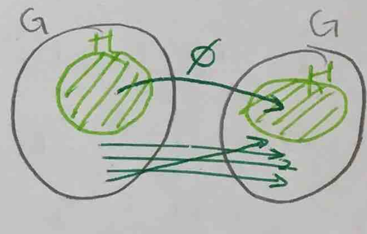
$\phi : D_6 \to D_6$

$\phi(x) = (sr)x(sr)^{-1}$



• For all $g \in G$, let $\phi(x) = gxg^{-1}$

② means you have this picture



Proof: we show ① ⇒ ② and ② ⇒ ③

① ⇒ ② Assume ① is true $[xhx^{-1} \in H \ \forall x \in G, h \in H]$

Let $g \in G$. We NTS that $gHg^{-1} = H$

By ① we have that $gHg^{-1} \subseteq H$

Let's now show $H \subseteq gHg^{-1}$, let $h \in H$

By ① we know that $g^{-1}Hg \in H$ [① with $x = g^{-1}$]

so $g^{-1}hg = h'$ where $h' \in H$, so, $h = gh'g^{-1} \in gHg^{-1}$

so, $gHg^{-1} = H$

② ⇒ ①

Suppose $gHg^{-1} = H \ \forall g \in G$ Then $\{ghg^{-1} \mid h \in H\} = H \ \forall g \in G$

so $ghg^{-1} \in H \ \forall g \in G, h \in H$ ∴ ① ⇒ ②

② ⇒ ③

Suppose $gHg^{-1} = H \ \forall g \in G$

let $g \in G$ we want to show that $gH = Hg$

$\underline{gH \subseteq Hg}$

Let $x \in gH$

Then $x = gh$ where $h \in H$

By assumption $xg^{-1} = \underbrace{ghg^{-1}}_{\text{in } gHg^{-1}} \in H$

so, $xg^{-1} = h'$ where $h' \in H$ Thus $x = h'g \in Hg$

$\underline{Hg \subseteq gH}$

same kind of proof

thus $gH = Hg \ \forall g \in G$

③ ⇒ ②

Suppose $gH = Hg \ \forall g \in G$.

we show ① is true in this case and therefore ② is true since ① ⇒ ②. Let $g \in G$ and $h \in H$

Then since $gh \in gH$ and $gH = Hg$ we know $gh = h'g$ where $h' \in H$ so $ghg^{-1} = h' \in H$

Thus $ghg^{-1} \in H \ \forall g \in G, h \in H$ so ① is true

Hence ② is true ▨