Hw # 7

(1)

element	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{2})$
order	1	2	3	6	3	6

$\langle(\bar{1},\bar{1})\rangle = \{(\bar{0},\bar{0}), (\bar{1},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{2})\} = \mathbb{Z}_2 \times \mathbb{Z}_3$

So, $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.

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(2)

$$\left.\begin{array}{l}(\bar{2},\bar{3}) \\ (\bar{2},\bar{3})+(\bar{2},\bar{3}) = (\bar{0},\bar{6}) \\ (\bar{2},\bar{3})+(\bar{2},\bar{3})+(\bar{2},\bar{3}) = (\bar{2},\bar{9}) \\ (\bar{2},\bar{3})+(\bar{2},\bar{3})+(\bar{2},\bar{3})+(\bar{2},\bar{3}) = (\bar{0},\bar{0})\end{array}\right\}$$

So the order of $(\bar{2},\bar{3})$ is 4.

Claim: Up to isomorphism, the only groups of size 4 are $\mathbb{Z}_4$ an $\mathbb{Z}_2 \times \mathbb{Z}_2$.

proof: Suppose $G = \{e, a, b, c\}$ is a group of order 4. case 1: If any of $a, b,$ or c has order 4, then G is cyclic and so is isomorphic to $\mathbb{Z}_4$. case 2: Otherwise, $a^2 = b^2 = c^2 = e$. This is enough to fill in the group table for G. For example: Claim: $ab = c$. pf of claim: Suppose $ab = e$. Then $a^{-1} = b$. But $a^{-1} = a$ since $a^2 = e$. So, $ab \neq e$. Suppose $ab = a$. Then $b = e$. So, $ab \neq a$. Similarly $ab \neq b$. ∎

Here are the other products:

$ab = c$
$ac = b$

$ba = c$
$bc = a$

$ca = b$
$cb = a$

G	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Compare this to

$\mathbb{Z}_2 \times \mathbb{Z}_2$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

(4) We need to find cyclic subgroups of size 4 and also subgroups of size 4 that are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\begin{array}{lcccccccc}
\mathbb{Z}_2 \times \mathbb{Z}_4 = \{ & (\bar{0},\bar{0}), & (\bar{0},\bar{1}), & (\bar{0},\bar{2}), & (\bar{0},\bar{3}), & (\bar{1},\bar{0}), & (\bar{1},\bar{1}), & (\bar{1},\bar{2}), & (\bar{1},\bar{3}) \} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\underline{\text{order:}} & 1 & 4 & 2 & 4 & 2 & 4 & 2 & 4
\end{array}$$

<u>Subgroups of size 4 that are cyclic:</u>

$$\langle (\bar{0},\bar{1}) \rangle = \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}) \} = \langle (\bar{0},\bar{3}) \rangle$$

$$\langle (\bar{1},\bar{1}) \rangle = \{ (\bar{0},\bar{0}), (\bar{1},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{3}) \} = \langle (\bar{1},\bar{3}) \rangle$$

<u>non-cyclic subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$:</u>

$$H = \{ (\bar{0},\bar{0}), (\bar{0},\bar{2}), (\bar{1},\bar{0}), (\bar{1},\bar{2}) \}$$

(5) If $G \times G$ is cyclic then there exists $(g,h) \in G \times G$ such that $G \times G = \langle (g,h) \rangle$. Let $x \in G$. Then $(x,x) \in G \times G$. Hence, $(x,x) = (g,h)^k = (g^k, h^k)$ for some integer k. Hence $x = g^k$. So, $G = \langle g \rangle$.

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(6) Let $(x,y), (a,b) \in G \times H$. Then $xa = ax$ and $yb = by$ since G and H are abelian. Therefore,

$$(x,y)(a,b) = (xa, yb) = (ax, by) = (a,b)(x,y).$$

So, $G \times H$ is abelian.

⑦ Since $G_1 \cong G_2$ and $H_1 \cong H_2$ there exist isomorphisms $\varphi_G: G_1 \to G_2$ and $\varphi_H: H_1 \to H_2$.

Let $\varphi: G_1 \times H_1 \to G_2 \times H_2$ be defined by $\varphi(g,h) = (\varphi_G(g), \varphi_H(h))$.

<u>φ is a homomorphism:</u> Let $(g,h),(a,b) \in G_1 \times H_1$.

Then,

$$\varphi((g,h)(a,b)) = \varphi(ga, hb) = (\varphi_G(ga), \varphi_H(hb))$$

$$\underset{\boxed{\varphi_G \text{ and } \varphi_H \text{ are homs}}}{=} (\varphi_G(g)\varphi_G(a), \varphi_H(h)\varphi_H(b)) = (\varphi_G(g), \varphi_H(h))(\varphi_G(a), \varphi_H(b))$$

$$= \varphi(g,h)\varphi(a,b).$$

<u>φ is 1-1:</u> Suppose $\varphi(g,h) = \varphi(a,b)$ for some $(g,h),(a,b) \in G_1 \times H_1$. Then $(\varphi_G(g), \varphi_H(h)) = (\varphi_G(a), \varphi_H(b))$. So, $\varphi_G(g) = \varphi_G(a)$ and $\varphi_H(h) = \varphi_H(b)$. Since φ_G and φ_H are 1-1, $g = a$ and $h = b$. Hence $(g,h) = (a,b)$. So, φ is 1-1.

<u>φ is onto:</u> Let $(x,y) \in G_2 \times H_2$. Since φ_G is onto there exists $g \in G_1$ with $\varphi_G(g) = x$. Since φ_H is onto there exists $h \in H_1$ with $\varphi_H(h) = y$. Hence $\varphi(g,h) = (\varphi_G(g), \varphi_H(h)) = (x,y)$. So, φ is onto.