Hw #8

1 (a) Let $x, y \in \mathbb{Z}$. Then $\varphi(x+y) = 5(x+y) =$
$= 5x + 5y = \varphi(x) + \varphi(y)$. So, φ is a homomorphism.

Note that $\varphi(x) = 5x = 0$ iff $x = 0$.
Thus, $\ker(\varphi) = \{0\}$.

(b) Let $x, y \in \mathbb{R}$. Then $\varphi(x+y) = 2^{x+y} =$
$= 2^x 2^y = \varphi(x)\varphi(y)$, So, φ is a homomorphism.

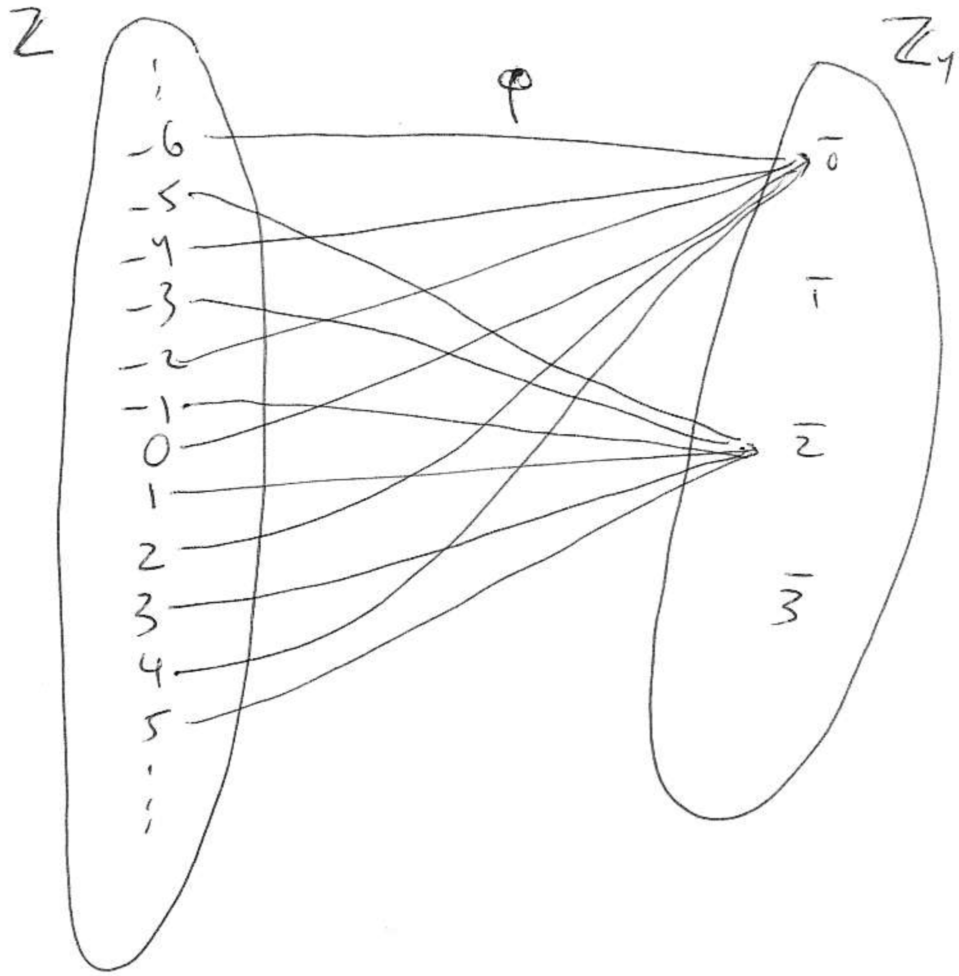
$2^x = 1$ iff $x = 0$. Thus $\ker(\varphi) = \{0\}$.

(c) Let $g, h \in G$. Since G is abelian
$\varphi(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = \varphi(g)\varphi(h)$.
Hence φ is a homomorphism.

Note that $\varphi(g) = g^{-1} = e$ iff $g = e$.
Hence $\ker(\varphi) = \{e\}$.

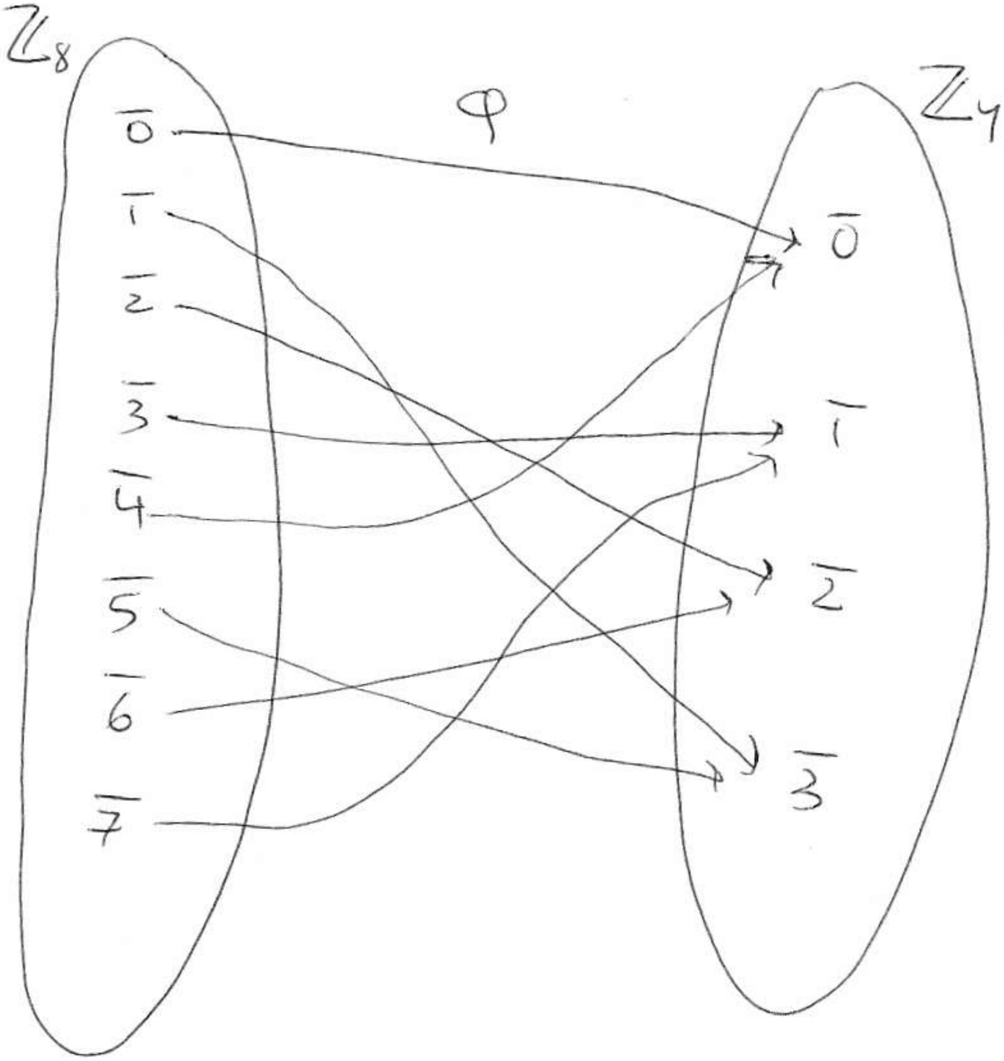
(2) $\varphi(3) = \varphi(1+1+1) = \varphi(1) + \varphi(1) + \varphi(1)$
$= \bar{2} + \bar{2} + \bar{2} = \bar{2}$

$\varphi(-1)$ is the inverse of $\varphi(1)$. Hence
$\varphi(-1) = \bar{2}$. Thus $\varphi(-2) = \varphi((-1)+(-1)) = \bar{2} + \bar{2} = \bar{0}$



$\ker(\varphi) = 2\mathbb{Z}$
$\varphi(\mathbb{Z}) = \{\bar{0}, \bar{2}\}$

(3)



$\mathrm{Ker}(\varphi) = \{\bar{0}, \bar{4}\}$

$\varphi(\mathbb{Z}_8) = \mathbb{Z}_4$

(4) Let $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_5$ be a homomorphism. $\overline{1} \in \mathbb{Z}_{12}$ has order 12. Thus $\varphi(\overline{1})$ must have order dividing 12.

element	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
order	$\overline{1}$	$\overline{5}$	$\overline{5}$	$\overline{5}$	$\overline{5}$

} in $\mathbb{Z}_5$

Hence $\varphi(\overline{1}) = \overline{0}$. Thus, $\varphi(\overline{x}) = \overline{0}$ for all $\overline{x} \in \mathbb{Z}_{12}$. So, φ is trivial.

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(5) Suppose $\varphi : \mathbb{Z}_3 \to \mathbb{Z}$ is a homomorphism. Then $\varphi(\overline{1})$ must be an element of $\mathbb{Z}$ with order dividing 3. But all the elements of $\mathbb{Z}$ have infinite order except for 0 which ha order 1. Thus, $\varphi(\overline{1}) = 0$. So, $\varphi(\overline{2}) = \varphi(\overline{1}) + \varphi(\overline{1}) = 0$. Also, $\varphi(\overline{0}) = 0$. Hence φ is the trivial map.

(6) Let $|G| = p$ be prime. Since $\ker(\varphi)$ is a subgroup of G, $|\ker(\varphi)|$ divides p. Since p is prime, either $|\ker(\varphi)| = 1$ or $|\ker(\varphi)| = p$.

If $|\ker(\varphi)| = 1$, then $\ker(\varphi) = \{e\}$. Hence φ is 1-1 by thm in class.

If $|\ker(\varphi)| = p$, then $\ker(\varphi) = G$. Hence $\varphi(x) = e'$ for all $x \in G$. So, φ is trivial.

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(7)

($\Rightarrow$) Suppose that $\varphi(G)$ is abelian. Let $x, y \in G$. Then

$$\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1}$$
$$\underset{\boxed{\varphi(G) \text{ is abelian}}}{=} \varphi(x)\varphi(x)^{-1}\varphi(y)\varphi(y)^{-1} = e'.$$

Hence $xyx^{-1}y^{-1} \in \ker(\varphi)$.

($\Leftarrow$) Let $a, b \in \varphi(G)$. Then $a = \varphi(x)$ and $b = \varphi(y)$ for some $x, y \in G$. Hence

$$ab a^{-1}b^{-1} = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = \varphi(xyx^{-1}y^{-1}) \underset{\boxed{\text{since } xyx^{-1}y^{-1} \in \ker(\varphi) \text{ by assumption}}}{=} e'$$

Hence $ab = ba$. So, $\varphi(G)$ is abelian.