Homework 2 Solutions

(1)

$\mathbb{Z}_3^\times = \{\bar{1}, \bar{2}\}$

$\mathbb{Z}_4^\times = \{\bar{1}, \bar{3}\}$

$\mathbb{Z}_5^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$\mathbb{Z}_6^\times = \{\bar{1}, \bar{5}\}$

$\mathbb{Z}_7^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$

$\mathbb{Z}_8^\times = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$

$\mathbb{Z}_9^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$

$\mathbb{Z}_{10}^\times = \{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$

$\mathbb{Z}_{11}^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}\}$

$\mathbb{Z}_{12}^\times = \{\bar{1}, \bar{5}, \bar{7}, \overline{11}\}$

$\mathbb{Z}_{13}^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}, \overline{12}\}$

$\mathbb{Z}_{14}^\times = \{\bar{1}, \bar{3}, \bar{5}, \bar{9}, \overline{11}, \overline{13}\}$

(2)

(a) If $n \geq 2$ then

$$n\mathbb{Z} = \{\ldots, -2n, -n, 0, n, 2n, 3n, \ldots\}$$

is not an integral domain since $1 \notin n\mathbb{Z}$.

(b) $\mathbb{Z} \times \mathbb{Z}$ has zero divisors so it is not an integral domain:

$(0,0)$ is the additive identity.

$$\underset{\substack{\uparrow \\ \text{not} \\ \cancel{\text{zero}} \\ \text{additive} \\ \text{identity}}}{(1,0)} \cdot \underset{\substack{\uparrow \\ \text{not} \\ \text{additive} \\ \text{identity}}}{(0,1)} = (0,0)$$

(c) Same as (b). For example

$$(\bar{1},\bar{0}) \cdot (\bar{0},\bar{1}) = (\bar{0},\bar{0})$$

So, $\mathbb{Z}_2 \times \mathbb{Z}_3$ has zero divisors.

(d) $\mathbb{Z}_5$ is an integral domain since 5 is prime.

(e) $\mathbb{Z}_{106}$ is not an integral domain since 106 is not prime. For example,

$$\underbrace{\overline{2} \cdot \overline{53}}_{\text{not zero}} = \overline{106} = \underset{\substack{\uparrow \\ \text{zero}}}{\overline{0}}$$

(f) $M_2(\mathbb{R})$ is not an integral domain

The additive identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

And

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

not additive identity

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(3) Since R_1 and R_2 are integral domains they have mult. identities 1_1 and 1_2. Let 0_1 and 0_2 be the additive identities of R_1 and R_2. Then $(0_1, 0_2)$ is the additive identity of $R_1 \times R_2$. Note that $(1_1, 0_2) \neq (0_1, 0_2)$ and $(0_1, 1_2) \neq (0_1, 0_2)$ but

$$(1_1, 0_2) \cdot (0_1, 1_2) = (0_1, 0_2).$$

So $R_1 \times R_2$ has zero divisors, and is not an integral domain.

(4) (a) S is a commutative ring with identity since S is a subring of R and $1 \in S$. S has no zero divisors since if it did then R would also have zero divisors since $S \subseteq R$. Thus, S is an integral domain.

(b) In this case, S will not be an integral domain because $1 \notin S$.

Ex: $\mathbb{Z}$ is an integral domain.
$2\mathbb{Z}$ is a subring of $\mathbb{Z}$ $(2\mathbb{Z} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\})$
$2\mathbb{Z}$ is not an integral domain since $1 \notin 2\mathbb{Z}$

[Subdomain means a subring that is an integral domain.]

(5) We know from hw #1 that $R \cap S$ is a subring of T.

- Since R and S are both subdomains of T we know that $1 \in R$ and $1 \in S$. Hence $1 \in R \cap S$.
- Since R and S are both subdomains of T ~~we know that they are both~~ and T is commutative, we know that $R \cap S$ is commutative.
- What about zero divisors. Suppose that $x \in R \cap S$ is a zero divisor in $R \cap S$. Then $x \neq 0$ and there exists $y \neq 0$ with $y \in R \cap S$ and $xy = 0$. But then $x \in R$ and $y \in R$ and $xy = 0$. This would say that R has zero divisors, which isn't true. Thus, $R \cap S$ has no zero divisors.
- So, $R \cap S$ is an integral domain.

(6) Suppose that $x \in R$ is an idempotent and R is an integral domain.

Then $x \cdot x = x$.

So, $x \cdot x - x = 0$.

So, $x(x-1) = 0$.

Since R is an integral domain, either

$x = 0$ or $x - 1 = 0$.

Thus, $x = 0$ or $x = 1$.

So, the only idempotent elements of an integral domain are 0 and 1.