Homework #3 Solutions

① (a) In $\mathbb{Z}_3[x]$ we have

$$f(x)+g(x) = (\bar{2}x^2+x)+(x^2+\bar{2}x+\bar{1})$$
$$= \bar{3}x^2+\bar{3}x+\bar{1} = \bar{1}$$

$$f(x)\cdot g(x) = (\bar{2}x^2+x)(x^2+\bar{2}x+\bar{1})$$
$$= \bar{2}x^4+\bar{4}x^3+\bar{2}x^2+x^3+\bar{2}x^2+x$$
$$= \bar{2}x^4+\bar{2}x^3+x^2+x$$

(b) In $\mathbb{Z}_2[x]$ we have

$$f(x)+g(x) = (x^3+x^2+x+\bar{1})+(x^2+\bar{1})$$
$$= x^3+x$$

$$f(x)\cdot g(x) = (x^3+x^2+x+\bar{1})(x^2+\bar{1})$$
$$= x^5+x^3+x^4+x^2+x^3+x+x^2+\bar{1}$$
$$= x^5+x^4+x+\bar{1}$$

(2) $\bar{0}, \bar{1}, \bar{2}, x, x+\bar{1}, x+\bar{2}, \bar{2}x, \bar{2}x+\bar{1}, \bar{2}x+\bar{2}$

(3) $\bar{0}, \bar{1}, x, x+\bar{1}, x^2, x^2+\bar{1}, x^2+x+\bar{1}, x^2+x$

(4) $f(\bar{0}) = 0^2+\bar{1} = \bar{1} \neq \bar{0}$

$f(\bar{1}) = \bar{1}^2+\bar{1} = \bar{2} = \bar{0}$

zeros of $f(x)=x^2+\bar{1}$ in $\mathbb{Z}_2[x]$ are $x=\bar{1}$.

(5) $f(\bar{0}) = \bar{0}^2+\bar{2} = \bar{2} \neq \bar{0}$

$f(\bar{1}) = \bar{1}^2+\bar{2} = \bar{3} = \bar{0}$

$f(\bar{2}) = \bar{2}^2+\bar{2} = \bar{6} = \bar{0}$

zeros of $f(x)=x^2+\bar{2}$ in $\mathbb{Z}_3[x]$ are $x=\bar{1}, \bar{2}$

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(a) Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
and $q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$
be elements of $R[x]$ with $a_n \neq 0$ and $b_m \neq 0$.
So, $\deg(p) = n$ and $\deg(q) = m$. and $n, m \geq 0$.
Since R is an integral domain
we know that $a_n b_m \neq 0$.

Thus,

$$p(x) \cdot q(x) = a_n b_m x^{n+m} + (a_n b_{m-1} + a_{n-1} b_m) x^{n+m-1} + \dots + a_0 b_0$$

has degree $n+m$.

(b) Let $p(x), q(x) \in R[x]$
and assume $p(x) \neq 0$ and $q(x) \neq 0$.
Then, $p(x) = a_n x^n + \dots + a_0$
and $q(x) = b_m x^m + \dots + b_0$ with
$a_n \neq 0$ and $b_m \neq 0$. Thus, $a_n b_m \neq 0$ since R is an integral domain.

So, $p(x) \cdot q(x) = a_n b_m x^{n+m} + (a_n b_{m-1} + a_{n-1} b_m) x^{n+m-1} + \dots + a_0 b_0 \neq 0$
since $a_n b_m \neq 0$. So, $R[x]$ is
an integral domain.

(c) Suppose that $p(x) \in R[x]$ is a unit. Then there exists $q(x) \in R[x]$ with $p(x) \cdot q(x) = 1$. By (a)

$$\deg(p) + \deg(q) = \deg(1) = 0.$$

Thus, $\deg(p) = 0$ and $\deg(q) = 0$.

So, $p(x) = a_0$ and $q(x) = b_0$ where $a_0, b_0 \in R$. And $a_0 b_0 = 1$. Thus, $p(x)$ is a unit of R, or $p(x) \in R^{\times}$.