Math 456
Homework # 5 - Ideals

1. List 5 elements from the ideal $\langle x^2 + \overline{1} \rangle$ in $\mathbb{Z}_3[x]$.

2. Which of the following are ideals of $\mathbb{Z} \times \mathbb{Z}$?

   (a) $\{(a, a) \mid a \in \mathbb{Z}\}$

   (b) $\{(2a, 3b) \mid a, b \in \mathbb{Z}\}$

   (c) $\{(a, 0) \mid a \in \mathbb{Z}\}$

   (d) $\{(a, -a) \mid a \in \mathbb{Z}\}$

3. Prove that $I = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{0}, \overline{2})\}$ is an ideal of the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_3$.

4.

   (a) Prove that every ideal of $\mathbb{Z}_n$ is principal. That is each ideal is of the form
   $$\langle \overline{k} \rangle = \{\overline{0}, \overline{k}, \overline{2k}, \overline{3k}, \ldots, \overline{(n-1)k}\}$$
   where $\overline{k} \in \mathbb{Z}_n$.

   (b) Find all the ideals of $\mathbb{Z}_6$.

   (c) Find all the ideals of $\mathbb{Z}_8$.

   (d) Calculate the ideals $\langle \overline{13} \rangle$ and $\langle \overline{2} \rangle$ of $\mathbb{Z}_{26}$.

5. Determine which of the sets below is an ideals of $M_2(\mathbb{R})$.

   (a)
   $$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

(b)

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

6. Let R and R' be rings. Let $\phi : R \to R'$ be a ring homomorphism.

   (a) Prove that $\ker(\phi)$ is a an ideal of R.

   (b) Suppose that ϕ is onto. Prove that $\phi(R) = \{\phi(x) \mid x \in R\}$ is an ideal of R'.

7. Let R be a ring with additive identity denoted by 0. Show that $\{0\}$ and R are ideals of R.

8. Let I be an ideal of a ring R. Show that I is a subring of R.

9. Let R be a commutative ring with additive identity 0 and multiplicative identity 1 with $1 \neq 0$. Let $a \in R$. Prove that $\langle a \rangle = \{ra \mid r \in R\}$ is an ideal of R.

10. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Show that

$$I = \{(\overline{0}, \overline{0}), (\overline{2}, \overline{0}), (\overline{0}, \overline{2}), (\overline{2}, \overline{2})\}$$

is a principal ideal of R.