Math 456
Homework # 6 - Quotient Rings

1. Calculate the elements of the factor rings R/I and calculate their addition and multiplication tables.

(a) $R = \mathbb{Z}$ and $I = 3\mathbb{Z}$.

(b) $R = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $I = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{0}, \overline{2})\}$

(c) $R = \mathbb{Z}_8$ and $I = \langle \overline{4} \rangle = \{\overline{0}, \overline{4}\}$.

2. Let R be a ring, I be an ideal, $x \in R$, and $n \in \mathbb{Z}$ with $n \geq 1$. Prove that $(x + I)^n = x^n + I$ in the quotient ring R/I.

3. Let $\phi : R \to R'$ be a ring homomorphism. Let I be an ideal of R. Prove that

$$\phi(I) = \{\phi(x) \mid x \in I\}$$

is an ideal of $\phi(R) = \{\phi(x) \mid x \in R\}$. In particular, if ϕ is onto, then $\phi(I)$ is an ideal of R'.

4. Let R be a ring. Prove that $R/\{0\}$ is isomorphic to R.

5. Let R be a ring and I be an ideal of R. We know that R/I is a ring. Prove the following:

(a) If R is commutative, then R/I is commutative.

(b) If R has a multiplicative identity that is denoted by 1, then $1 + I$ is a multiplicative identity for R/I.

6. Let R be a ring and I be an ideal of R. Let $\pi : R \to R/I$ be defined by $\pi(x) = x + I$. Prove that π is a ring homomorphism. (π is sometimes called the reduction homomorphism or canonical homomorphism.)

7. Let $\phi : R \to R'$ be a ring homomorphism. Let I' be an ideal of R'. Prove that

$$\phi^{-1}(I') = \{x \in R \mid \phi(x) \in I'\}$$

is an ideal of R.