Homework #6 Solutions

1. (a)

$0+3\mathbb{Z} = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$1+3\mathbb{Z} = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$2+3\mathbb{Z} = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

$\mathbb{Z}/3\mathbb{Z} = \{0+3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\}$

$\mathbb{Z}/3\mathbb{Z}, +$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$0+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$1+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$	$0+3\mathbb{Z}$
$2+3\mathbb{Z}$	$2+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$

$\mathbb{Z}/3\mathbb{Z}, \cdot$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$0+3\mathbb{Z}$	$0+3\mathbb{Z}$	$0+3\mathbb{Z}$	$0+3\mathbb{Z}$
$1+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$2+3\mathbb{Z}$	$0+3\mathbb{Z}$	$2+3\mathbb{Z}$	$1+3\mathbb{Z}$

(b) $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(\bar{0},\bar{0}), (\bar{1},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{2})\}$

$(\bar{0},\bar{0}) + I = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2})\}$

$(\bar{1},\bar{0}) + I = \{(\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2})\}$

$\mathbb{Z}_2 \times \mathbb{Z}_3 / I$, $+$	$(\bar{0},\bar{0})+I$	$(\bar{1},\bar{0})+I$
$(\bar{0},\bar{0})+I$	$(\bar{0},\bar{0})+I$	$(\bar{1},\bar{0})+I$
$(\bar{1},\bar{0})+I$	$(\bar{1},\bar{0})+I$	$(\bar{0},\bar{0})+I$

$\mathbb{Z}_2 \times \mathbb{Z}_3 / I$, $\cdot$	$(\bar{0},\bar{0})+I$	$(\bar{1},\bar{0})+I$
$(\bar{0},\bar{0})+I$	$(\bar{0},\bar{0})+I$	$(\bar{0},\bar{0})+I$
$(\bar{1},\bar{0})+I$	$(\bar{0},\bar{0})+I$	$(\bar{1},\bar{0})+I$

(c) $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

$I = \langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$

$\bar{0} + I = \{\bar{0}, \bar{4}\} = \bar{4} + I$

$\bar{1} + I = \{\bar{1}, \bar{5}\} = \bar{5} + I$

$\bar{2} + I = \{\bar{2}, \bar{6}\} = \bar{6} + I$

$\bar{3} + I = \{\bar{3}, \bar{7}\} = \bar{7} + I$

$\mathbb{Z}_8/I, +$	$\bar{0}+I$	$\bar{1}+I$	$\bar{2}+I$	$\bar{3}+I$
$\bar{0}+I$	$\bar{0}+I$	$\bar{1}+I$	$\bar{2}+I$	$\bar{3}+I$
$\bar{1}+I$	$\bar{1}+I$	$\bar{2}+I$	$\bar{3}+I$	$\bar{0}+I$
$\bar{2}+I$	$\bar{2}+I$	$\bar{3}+I$	$\bar{0}+I$	$\bar{1}+I$
$\bar{3}+I$	$\bar{3}+I$	$\bar{0}+I$	$\bar{1}+I$	$\bar{2}+I$

$\mathbb{Z}_8/I, \cdot$	$\bar{0}+I$	$\bar{1}+I$	$\bar{2}+I$	$\bar{3}+I$
$\bar{0}+I$	$\bar{0}+I$	$\bar{0}+I$	$\bar{0}+I$	$\bar{0}+I$
$\bar{1}+I$	$\bar{0}+I$	$\bar{1}+I$	$\bar{2}+I$	$\bar{3}+I$
$\bar{2}+I$	$\bar{0}+I$	$\bar{2}+I$	$\bar{0}+I$	$\bar{2}+I$
$\bar{3}+I$	$\bar{0}+I$	$\bar{3}+I$	$\bar{2}+I$	$\bar{1}+I$

② We prove this by induction. When $n=1$, $(x+I)^1 = x^1 + I$. Let $k \geq 1$. Suppose that $(x+I)^k = x^k + I$. Then

$$(x+I)^{k+1} = (x+I)^k (x+I) = (x^k + I)(x+I) = x^{k+1} + I$$

Hence $(x+I)^n = x^n + I$ for all $n \geq 1$.

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③ Let 0 and $0'$ be the additive identities of R and R'.

- We know that $\varphi(0) = 0'$. Since I is an ideal of R we know that $0 \in I$. Hence $0' \in \varphi(I)$.
- Let $x, y \in \varphi(I)$. Then $\varphi(a) = x$ and $\varphi(b) = y$ where $a, b \in I$. Since I is an ideal $a - b \in I$. Hence
$$x - y = \varphi(a) - \varphi(b) = \varphi(a-b) \in \varphi(I).$$
- Let $z \in \varphi(I)$ and $r' \in \varphi(R)$. Then $z = \varphi(c)$ and $r' = \varphi(r)$ where $c \in I$ and $r \in R$. Since I is an ideal $rc \in I$ and $cr \in I$. Hence,
$$r'z = \varphi(r)\varphi(c) = \varphi(rc) \in \varphi(I)$$
and $zr' = \varphi(c)\varphi(r) = \varphi(cr) \in \varphi(I)$.

(4) Define the map $f: R \to R$ where $f(x) = x$ for all $x \in R$. Then f is a ring homomorphism since

$$f(a+b) = a+b = f(a) + f(b)$$

and $f(ab) = ab = f(a)f(b)$

for all $a, b \in R$.

Note that $\ker(f) = \{0\}$. and $f(R) = R$.
Hence by the 1st homomorphism theorem, $R/\{0\} = R/\ker(f) \cong f(R) = R$.

(5)

(a) Let $x, y \in R$. Then $xy = yx$ since R is commutative. Hence

$$(x+I)(y+I) = xy + I = yx + I$$
$$= (y+I)(x+I).$$

Hence R/I is commutative.

(b) Let $x \in R$. Then

$$(1+I)(x+I) = 1 \cdot x + I = x + I$$

and $(x+I)(1+I) = x \cdot 1 + I = x + I.$

Hence $1+I$ is a multiplicative identity for R/I.

(6) Let $x, y \in R$. Then

$$\pi(x+y) = (x+y) + I = (x+I) + (y+I)$$
$$= \pi(x) + \pi(y).$$

and

$$\pi(xy) = xy + I = (x+I)(y+I)$$
$$= \pi(x)\pi(y).$$

So, π is a ring homomorphism.

(7) Let 0 and $0'$ be the additive identities of R and R'.

- Note that $0' \in I'$ since I' is an ideal of R'.

Since $\varphi(0) = 0' \in I'$ we have that $0 \in \varphi^{-1}(I')$.

- Let $x, y \in \varphi^{-1}(I')$. Thus, $\varphi(x) \in I'$ and $\varphi(y) \in I'$. Since I' is an ideal we know that $\varphi(x) - \varphi(y) = \varphi(x-y) \in I'$. Hence $x - y \in \varphi^{-1}(I')$.

- Let $x \in \varphi^{-1}(I')$ and $r \in R$. Thus, $\varphi(x) \in I'$. Since I' is an ideal of R' and $\varphi(r) \in R'$ we know that

$$\varphi(xr) = \varphi(x)\varphi(r) \in I'$$

and $$\varphi(rx) = \varphi(r)\varphi(x) \in I'.$$