Homework 7 Solutions

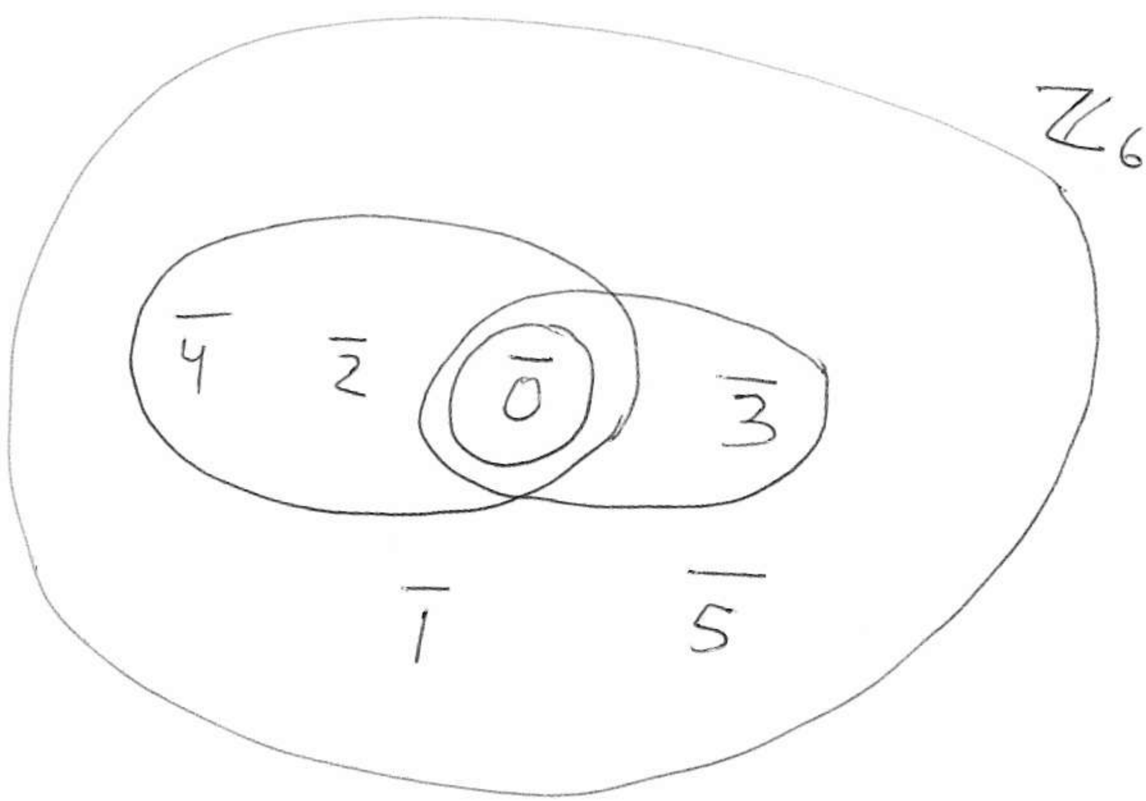
① The ideals of $\mathbb{Z}_6$ are

$\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} = \langle \bar{5} \rangle$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}\} = \langle \bar{4} \rangle$

$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$



The max ideals are $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$.
Since max ideals are prime, $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are prime.
$\{\bar{0}\}$ is not prime since $\bar{2} \cdot \bar{3} \in \{\bar{0}\}$ but $\bar{2} \notin \{\bar{0}\}$ and

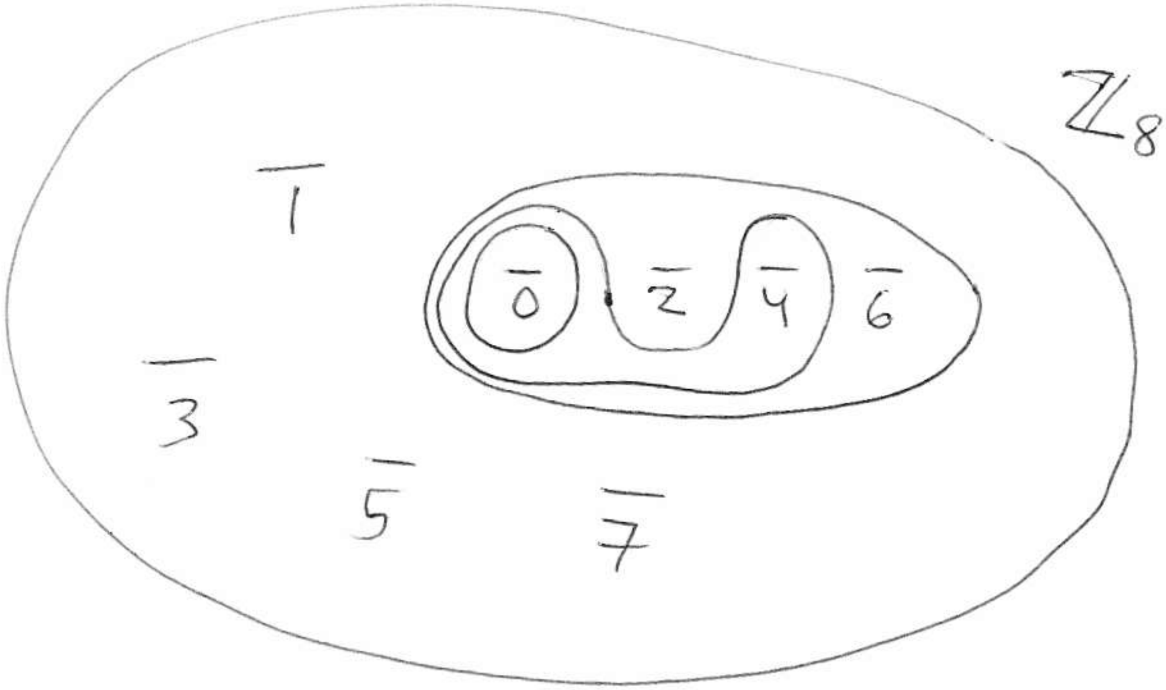
② The ideals of $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ are

$\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\} = \langle \bar{7} \rangle = \langle \bar{3} \rangle = \langle \bar{5} \rangle$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6} \rangle = \langle \bar{6} \rangle$

$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$.

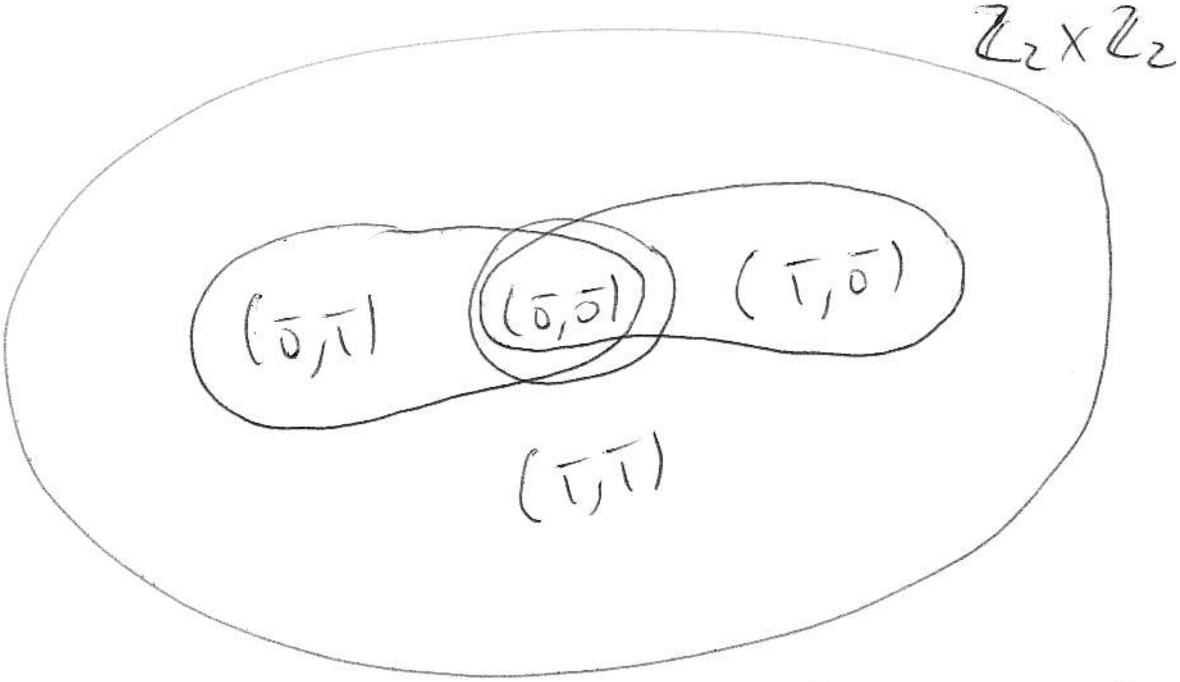


The only maximal ideal is $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.
This ideal is also prime since maximal ideals are also prime ideals.

$\{\bar{0}, \bar{4}\}$ is not prime since $\bar{2} \cdot \bar{2} \in \{\bar{0}, \bar{4}\}$ but $\bar{2} \notin \{\bar{0}, \bar{4}\}$.

$\{\bar{0}\}$ is not prime since $\bar{4} \cdot \bar{4} \in \{\bar{0}\}$ but $\bar{4} \notin \{\bar{0}\}$.

3. One can show that the ideals of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\bar{0},\bar{0}),(\bar{1},\bar{0}),(\bar{0},\bar{1}),(\bar{1},\bar{1})\}$ are the following: $\{(\bar{0},\bar{0})\}$, $\{(\bar{0},\bar{0}),(\bar{0},\bar{1})\}$, and $\{(\bar{0},\bar{0}),(\bar{1},\bar{0})\}$.



Thus, $\{(\bar{0},\bar{0}),(\bar{1},\bar{0})\}$ and $\{(\bar{0},\bar{0}),(\bar{0},\bar{1})\}$ are maximal ideals and also prime ideals.
$\{(\bar{0},\bar{0})\}$ is not prime since $(\bar{0},\bar{1})\cdot(\bar{1},\bar{0}) \in \{(\bar{0},\bar{0})\}$ but $(\bar{0},\bar{1}) \notin \{(\bar{0},\bar{0})\}$ and $(\bar{1},\bar{0}) \notin \{(\bar{0},\bar{0})\}$.

④ $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$
Which is not a field. So, $6\mathbb{Z}$ is not maximal. $\mathbb{Z}_6$ is not an integral domain, hence $6\mathbb{Z}$ is not a prime ideal.

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⑤ We know that $R/\{0\} \cong R$.
Hence $R/\{0\}$ is an integral domain.
So, $\{0\}$ is a prime ideal.

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⑥ Since I is maximal, R/I is a field. Hence R/I is an integral domain. Hence I is a prime ideal.

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⑦ $\{0\}$ is a prime ideal of $\mathbb{Z}$ since $\mathbb{Z}$ is an integral domain. $\{0\}$ is not maximal since $\{0\} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$.