Math 4570 - Homework # 4

The Matrix of a Linear Transformation

1. Consider the vector space $V = \mathbb{R}^3$ over the field $F = \mathbb{R}$. Let $\beta = [(1,0,0),(0,1,0),(0,0,1)]$ and $\beta' = [(1,0,1),(1,2,1),(0,0,1)]$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a,b,c) = (a+b, c, -a)$.

   (a) Calculate $[T]_\beta$

   (b) Calculate $[T]_{\beta'}$

   (c) Calculate $[T]_{\beta'}^{\beta}$

   (d) Let $x = (1,-2,4)$. Calculate $[x]_\beta$ and $[x]_{\beta'}$

   (e) Verify that $[T(x)]_\beta = [T]_\beta [x]_\beta$

   (f) Verify that $[T(x)]_{\beta'} = [T]_{\beta'} [x]_{\beta'}$

   (g) Verify that $[T(x)]_\beta = [T]_{\beta'}^{\beta} [x]_{\beta'}$

   (h) Calculate the change of coordinate matrix $[I]_{\beta'}^{\beta}$.

   (i) Use x from above and show that $[I]_{\beta'}^{\beta}[x]_{\beta'} = [x]_\beta$

   (j) Calculate the change of coordinate matrix $[I]_{\beta}^{\beta'}$.

   (k) Use x from above and show that $[I]_{\beta}^{\beta'}[x]_\beta = [x]_{\beta'}$

   (l) Show that $([I]_{\beta'}^{\beta})^{-1} = [I]_{\beta}^{\beta'}$

   (m) Show that $[T]_{\beta'}^{\beta}[I]_{\beta}^{\beta'} = [T]_\beta$

   (n) Show that $[T]_{\beta'} = [I]_{\beta}^{\beta'}[T]_\beta[I]_{\beta'}^{\beta}$

2. Let V and W be finite-dimensional vector spaces over a field F. Let β and γ be ordered bases for V and W respectively. Let $T : V \to W$ and $S : V \to W$ be linear transformations. Let $\alpha \in F$.

   (a) $[T+S]_\beta^\gamma = [T]_\beta^\gamma + [S]_\beta^\gamma$

   (b) $[\alpha T]_\beta^\gamma = \alpha [T]_\beta^\gamma$

3. Let V, W, and Z be finite-dimensional vector spaces over a field F with ordered bases α, β, and γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations.

(a) The composition $U \circ T : V \to Z$ is a linear transformation

(b) $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

4. Let V and W be finite-dimensional vector spaces over a field F with ordered bases α and β, respectively. Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations. If $[T_1]_\alpha^\beta = [T_2]_\alpha^\beta$, then $T_1 = T_2$.

5. Let F be a field. Let $A \in M_{n,n}(F)$ so that A is a square matrix. Recall that $L_A : F^n \to F^n$ given by left matrix multiplication $L_A(x) = Ax$ is a linear transformation.

   (a) Let $\beta = [v_1, v_2, \ldots, v_n]$ be the standard basis for F^n. That is, v_i is the vector with zeros in every spot except for a 1 in the i-th spot. Then $[L_A]_\beta = A$.

   (b) L_A is invertible iff A is invertible.

   (c) Let $\gamma = [w_1, w_2, \ldots, w_n]$ be any ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$ where Q is the $n \times n$ matrix whose i-th column is w_i, that is, $Q = (w_1|w_2|\cdots|w_n)$.