$\boxed{14.2}$

3) The splitting field of $(x^2-2)(x^2-3)(x^2-5)$ is $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ and $[E:\mathbb{Q}] = 8$.

The Galois group is given by the automorphisms

$$\sigma_1: \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_2: \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_3: \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases}$$

$$\sigma_4: \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases} \quad \sigma_5: \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_6: \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases}$$

$$\sigma_7: \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases} \quad \sigma_8: \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases}$$

$$G = \mathrm{Gal}(E/\mathbb{Q}) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}$$
$$= \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \underbrace{\sigma_2\sigma_3}_{\sigma_5}, \underbrace{\sigma_2\sigma_4}_{\sigma_6}, \underbrace{\sigma_3\sigma_4}_{\sigma_7}, \underbrace{\sigma_2\sigma_3\sigma_4}_{\sigma_8}\}$$
$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

Note: $E = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{5} + e\sqrt{2}\sqrt{3} + f\sqrt{2}\sqrt{5} + g\sqrt{3}\sqrt{5} + h\sqrt{2}\sqrt{3}\sqrt{5} \mid a, b, \dots, h \in \mathbb{Q}\}$

and if $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ then

$$\sigma(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{5} + e\sqrt{2}\sqrt{3} + f\sqrt{2}\sqrt{5} + g\sqrt{3}\sqrt{5} + h\sqrt{2}\sqrt{3}\sqrt{5})$$
$$= a + b(\sigma\sqrt{2}) + c(\sigma\sqrt{3}) + d(\sigma\sqrt{5}) + e(\sigma\sqrt{2})(\sigma\sqrt{3}) + f(\sigma\sqrt{2})(\sigma\sqrt{5}) + g(\sigma\sqrt{3})(\sigma\sqrt{5})$$
$$+ h(\sigma\sqrt{2})(\sigma\sqrt{3})(\sigma\sqrt{5})$$

Here are <u>some</u> of the subgroups and fixed fields

subgroup of G	fixed field
$\{\sigma_1\}$	$E = \mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$
$\{\sigma_1,\sigma_2\}$	$\mathbb{Q}(\sqrt{3},\sqrt{5})$
$\{\sigma_1,\sigma_3\}$	$\mathbb{Q}(\sqrt{2},\sqrt{5})$
$\{\sigma_1,\sigma_4\}$	$\mathbb{Q}(\sqrt{2},\sqrt{3})$
$\{\sigma_1,\sigma_5\}$	$\mathbb{Q}(\sqrt{6},\sqrt{5})$
$\{\sigma_1,\sigma_6\}$	$\mathbb{Q}(\sqrt{3},\sqrt{10})$
$\{\sigma_1,\sigma_7\}$	$\mathbb{Q}(\sqrt{2},\sqrt{15})$
$\{\sigma_1,\sigma_8\}$	$\mathbb{Q}(\sqrt{6},\sqrt{10},\sqrt{15})$
$\{\sigma_1,\sigma_2,\sigma_3,\sigma_2\sigma_3\}$	$\mathbb{Q}(\sqrt{5})$
$\{\sigma_1,\sigma_2,\sigma_4,\sigma_2\sigma_4\}$	$\mathbb{Q}(\sqrt{3})$
$\{\sigma_1,\sigma_3,\sigma_4,\sigma_3\sigma_4\}$	$\mathbb{Q}(\sqrt{2})$
$\vdots$	$\vdots$