$\boxed{9.2}$

(3) $F[x]/(f(x))$ is a field iff

$(f(x))$ is a maximal ideal. iff

$(f(x))$ is a prime ideal (because $F[x]$ is a PID -- use Prop 7 on page 280 and prop from class).

We finish the proof with the following facts:

Fact: $(f(x))$ is a prime ideal iff $f(x)$ is irreducible.

proof: ($\Rightarrow$) Suppose $(f(x))$ is a prime ideal. Suppose $f(x) = a(x)b(x)$. Then either $a(x) \in (f(x))$ or $b(x) \in (f(x))$. WLOG assume $a(x) \in f(x)$. Then $a(x) = f(x)g(x)$, where $g(x) \in F[x]$. So, $f(x) = a(x)b(x) = f(x)g(x)b(x)$. This implies that $g(x)$ and $b(x)$ are constants, which implies that f is irreducible.

($\Leftarrow$) Suppose $f(x)$ is irreducible. We show that $(f(x))$ is maximal which implies that $(f(x))$ is a prime ideal. Suppose that $(f(x)) \subseteq I \subseteq F[x]$ where I is an ideal of $F[x]$.

Since $F[x]$ is a PID, $I = (g(x))$ where $g(x) \in F[x]$. Since $f(x) \in (g(x))$ we have that

$f(x) = g(x)h(x)$.

Since $f(x)$ is irreducible, either $g(x)$ or $h(x)$ is a unit.

<u>case 1:</u> Suppose $g(x)$ is a unit. Then $(g(x)) = F[x]$ by Prop 9 on page 253.

<u>case 2:</u> Suppose $h(x) = \alpha \in F$ is a unit. Then $f(x) = \alpha g(x)$. We have that $(f(x)) \subseteq (g(x))$.

Let $k(x) \in (g(x))$. Then $k(x) = g(x)m(x)$ where $m(x) \in F[x]$. So,

$h(x) = \frac{1}{\alpha} f(x) m(x) \in (f(x))$. So, $(g(x)) \subseteq (f(x))$.

Thus, $(f(x)) = (g(x))$.

Hence, $(f(x))$ is a maximal ideal.

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(a) $\mathbb{Z}[x]/(2) = \{ f(x) + (2) \mid f(x) \in \mathbb{Z}[x] \}$

Note that $2x^n \in (2)$ for all $n \geq 0$.

Thus,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \overline{a_n} x^n + \overline{a_{n-1}} x^{n-1} + \dots + \overline{a_1} x + \overline{a_0}$$

where $\overline{a_i} = \begin{cases} 0 & \text{if } 2 \text{ divides } a_i \\ 1 & \text{if } 2 \text{ does not divide } a_i \end{cases}$

So,

$$\mathbb{Z}[x]/(2) = \left\{ f(x) + (2) \;\middle|\; \begin{array}{l} f(x) \in \mathbb{Z}[x] \text{ and the} \\ \text{coefficients of } f(x) \\ \text{are either } 0 \text{ or } 1 \end{array} \right\}.$$

(b) $\mathbb{Z}[x]/(x)$

Note that $a_n x^n + \dots + a_1 x \in (x)$ for all $a_n, \dots, a_1 \in \mathbb{Z}$ and $n \geq 1$. Thus,

$$\mathbb{Z}[x]/(x) = \{ n + (x) \mid n \in \mathbb{Z} \}.$$